

# SHARING OF A SET OF MEROMORPHIC FUNCTIONS AND MONTEL'S THEOREM

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**ABSTRACT.** In this paper we prove the result: Let  $\mathcal{F}$  be a family of meromorphic functions on a domain  $\Omega$  such that every pair of members of  $\mathcal{F}$  shares a set  $S := \{\psi_1(z), \psi_2(z), \psi_3(z)\}$  in  $\Omega$ , where  $\psi_j(z)$ ,  $j = 1, 2, 3$  is meromorphic in  $\Omega$ . If for every  $f \in \mathcal{F}$ ,  $f(z_0) \neq \psi_i(z_0)$  whenever  $\psi_i(z_0) = \psi_j(z_0)$  for  $i, j \in \{1, 2, 3\}$  ( $i \neq j$ ) and  $z_0 \in \Omega$ , then  $\mathcal{F}$  is normal in  $\Omega$ . This result generalizes a result of M.Fang and W.Hong [*Some results on normal family of meromorphic functions, Bull. Malays. Math. Sci. Soc. (2)23 (2000),143-151,*] and in particular, it generalizes the most celebrated theorem of Montel-the Montel's theorem.

## 1. Introduction and Main Results

A family  $\mathcal{F}$  of meromorphic functions defined on a domain  $\Omega \subseteq \overline{\mathbb{C}}$  is said to be normal in  $\Omega$  if every sequence of elements of  $\mathcal{F}$  contains a subsequence which converges locally uniformly in  $\Omega$  with respect to the spherical metric, to a meromorphic function or  $\infty$  (see [9]). Montel's theorem (see [8]) states that if each member of  $\mathcal{F}$  omits three distinct fixed values in  $\overline{\mathbb{C}}$ , then  $\mathcal{F}$  is a normal family in  $\Omega$ . During the last about hundred years Montel's theorem has undergone various extensions and generalizations. For example, (i) the omitted values are allowed to vary with each member of the family [2], (ii) the omitted values can be replaced by meromorphic functions [3] (iii) the omitted values are replaced by mutually avoiding continuous functions [1].

Recall that two nonconstant meromorphic functions  $f$  and  $g$  defined on  $\Omega$  are said to share a set  $S$  IM in  $\Omega$  provided  $\overline{E}_f(S) = \overline{E}_g(S)$ ; where  $S$  is a subset of distinct points in  $\overline{\mathbb{C}}$  and

$$\overline{E}_f(S) := \bigcup_{a \in S} \{z \in \Omega : f(z) = a\},$$

with each  $a$ -point in  $\overline{E}_f(S)$  being counted only once.

Recently, involving the sharing of values or functions or more generally the sets, various generalizations of Montel's theorem have been obtained (for example, see [3, 5, 7, 10, 11, 12, 13]. In particular, M.Fang and W.Hong [5] extended Montel's Theorem as follows:

**Theorem 1.1.** *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\Omega \subset \mathbb{C}$ . If, for each pair of functions  $f$  and  $g$  in  $\mathcal{F}$  share the set  $S = \{0, 1, \infty\}$ , then the family  $\mathcal{F}$  is normal in  $\Omega$ .*

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And J.Chang, M.Fang and L.Zalcman [3] proved the following generalization of Montel's criterion:

**Theorem 1.2.** *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\Omega$  and let  $a(z)$ ,  $b(z)$  and  $c(z)$  be distinct meromorphic functions in  $\Omega$ , one of which may be  $\infty$  identically. If, for all  $f \in \mathcal{F}$  and  $z \in \Omega$ ,  $f(z) \neq a(z)$ ,  $f(z) \neq b(z)$  and  $f(z) \neq c(z)$ , then the family  $\mathcal{F}$  is normal in  $\Omega$ .*

Also, S.Zeng and I.Lahiri [14] improved the result of Montel by considering shared set of two distinct values and proved the following result:

**Theorem 1.3.** *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\Omega$  and  $M$  be a positive number and  $S = \{a, b\}$ , where  $a, b$  are distinct elements of  $\overline{\mathbb{C}}$ . Further, suppose that (i) each pair of functions  $f, g \in \mathcal{F}$  share the set  $S$  in  $\Omega$ , (ii) there exists a  $c \in \mathbb{C} - \{a, b\}$  such that for each  $f \in \mathcal{F}$ ,  $|f'(z)| \leq M$  whenever  $f(z) = c$  in  $\Omega$ , and (iii) each  $f \in \mathcal{F}$  has no simple  $b$ -points in  $\Omega$ . Then  $\mathcal{F}$  is normal in  $\Omega$ .*

In this paper, we extend Theorem 1.1 by replacing the elements of the shared set  $S$  by distinct meromorphic functions and hence obtain another variation on Montel's theorem.

**Theorem 1.4.** *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\Omega$  and let  $\psi_1(z)$ ,  $\psi_2(z)$  and  $\psi_3(z)$  be distinct meromorphic functions in  $\Omega$  such that*  
*(i) every  $f, g \in \mathcal{F}$  share the set  $S := \{\psi_1(z), \psi_2(z), \psi_3(z)\}$  in  $\Omega$ ,*  
*(ii) for every  $f \in \mathcal{F}$ ,  $f(z_0) \neq \psi_i(z_0)$  whenever  $\psi_i(z_0) = \psi_j(z_0)$  for  $i, j \in \{1, 2, 3\}$  ( $i \neq j$ ) and  $z_0 \in \Omega$ .*  
*Then  $\mathcal{F}$  is normal in  $\Omega$ .*

**Remark 1.5.** If for every  $f \in \mathcal{F}$ ,  $f(z) \neq \psi_i(z)$  for  $i = 1, 2, 3$ . Then Theorem 1.4 reduces to Theorem 1.2.

As an illustration of Theorem 1.4, we have the following example.

**Example 1.6.** Consider the family  $\mathcal{F} = \{f_m : m \in \mathbb{N}\}$ , where

$$f_m(z) = \frac{e^z}{3m}$$

on the unit disk  $\mathbb{D}$  and let  $\psi_1(z) = 0$ ,  $\psi_2(z) = e^z$  and  $\psi_3(z) = e^z/2$ . Clearly, for every  $f, g \in \mathcal{F}$ ,  $f$  and  $g$  share the set

$$S = \left\{0, e^z, \frac{e^z}{2}\right\}$$

of distinct meromorphic functions and the family  $\mathcal{F}$  can easily be seen to be normal in  $\mathbb{D}$ .

The following example show that the conditions (ii) is essential in Theorem 1.4.

**Example 1.7.** Consider the family  $\mathcal{F} = \{f_m : m \in \mathbb{N}\}$ , where

$$f_m(z) = 2mz$$

on the unit disk  $\mathbb{D}$  and let  $\psi_1(z) = z$ ,  $\psi_2(z) = z/2$  and  $\psi_3(z) = z/3$ . Clearly, for every  $f, g \in \mathcal{F}$ ,  $f$  and  $g$  share the set

$$S = \left\{z, \frac{z}{2}, \frac{z}{3}\right\}$$

in  $\mathbb{D}$ . However, the family  $\mathcal{F}$  is not normal in  $\mathbb{D}$ . Note that  $f_m(0) = \psi_1(0) = \psi_2(0) = \psi_3(0)$ , showing that we cannot drop the condition (ii) in Theorem 1.4.

Finally, the following example shows that the cardinality of set  $S$  in Theorem 1.4 cannot be reduced.

**Example 1.8.** Consider the family  $\mathcal{F} = \{f_m : m \in \mathbb{N}\}$ , where

$$f_m(z) = \tan mz + z$$

on the unit disk  $\mathbb{D}$  and let  $\psi_1(z) = z + i$ ,  $\psi_2(z) = z - i$ . Clearly, for every  $f, g \in \mathcal{F}$ ,  $f$  and  $g$  share the set

$$S = \{z + i, z - i\}$$

and  $\psi_1(z) \neq \psi_2(z)$  in  $\mathbb{D}$ . However, the family  $\mathcal{F}$  is not normal in  $\mathbb{D}$ .

Further one can ask what can be said about normality of  $\mathcal{F}$  if  $f$  is replaced by  $f^{(k)}$  in Theorem 1.4. In this direction, we prove the following result:

**Theorem 1.9.** *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\Omega$ , all of whose zeros have multiplicity at least  $k + 1$ , where  $k$  is a positive integer. Let  $\psi_1(z)$ ,  $\psi_2(z)$  and  $\psi_3(z)$  be distinct meromorphic functions in  $\Omega$  such that*

- (i) *for every  $f, g \in \mathcal{F}$ ,  $f^{(k)}$  and  $g^{(k)}$  share the set  $S := \{\psi_1(z), \psi_2(z), \psi_3(z)\}$  in  $\Omega$ ,*
- (ii) *for every  $f \in \mathcal{F}$ ,  $f(z_0) \neq \psi_i(z_0)$  whenever  $\psi_i(z_0) = \psi_j(z_0)$  for  $i, j \in \{1, 2, 3\}$  ( $i \neq j$ ) and  $z_0 \in \Omega$ .*

*Then  $\mathcal{F}$  is normal in  $\Omega$ .*

**Example 1.10.** Consider the family  $\mathcal{F} = \{f_m : m \in \mathbb{N}\}$ , where

$$f_m(z) = mz^k$$

on the unit disk  $\mathbb{D}$  and let  $\psi_1(z) = 0$ ,  $\psi_2(z) = 1/2$  and  $\psi_3(z) = 1/3$ . Clearly, for every  $f, g \in \mathcal{F}$ ,  $f^{(k)}$  and  $g^{(k)}$  share the set

$$S = \left\{0, \frac{1}{2}, \frac{1}{3}\right\}$$

in  $\mathbb{D}$ . However, the family  $\mathcal{F}$  is not normal in  $\mathbb{D}$ . This shows that the condition in Theorem 1.9 that the zeros of functions in  $\mathcal{F}$  have multiplicity at least  $k + 1$  cannot be dropped.

## 2. Notations and Lemmas

For  $z_0 \in \mathbb{C}$  and  $r > 0$ , we denote  $\mathbb{D}$  the open unit disk,  $D_r(z_0) = \{z : |z - z_0| < r\}$  and  $D'_r(z_0) = \{z : 0 < |z - z_0| < r\}$ . To prove our result, we require the following lemmas.

**Lemma 2.1.** (Zalcman's lemma) [9] *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\Omega$ . If  $\mathcal{F}$  is not normal at a point  $z_0 \in \Omega$ , there exist a sequence of points  $\{z_n\} \in \Omega$  with  $z_n \rightarrow z_0$ , a sequence of positive numbers  $\rho_n \rightarrow 0$  and a sequence of functions  $f_n \in \mathcal{F}$  such that*

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta)$$

*converges locally uniformly with respect to the spherical metric to  $g(\zeta)$ , where  $g(\zeta)$  is a non-constant meromorphic function on  $\mathbb{C}$ .*

**Lemma 2.2.** [3] *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\mathbb{D}$  and let  $a$  and  $b$  be distinct functions holomorphic on  $\mathbb{D}$ . Suppose that, for any  $f \in \mathcal{F}$  and any  $z \in \mathbb{D}$ ,  $f(z) \neq a(z)$  and  $f(z) \neq b(z)$ . If  $\mathcal{F}$  is normal in  $\mathbb{D} - \{0\}$ , then  $\mathcal{F}$  is normal in  $\mathbb{D}$ .*

**Lemma 2.3.** [4] *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\Omega$ , all of whose zeros have multiplicity at least  $k + 1$ , where  $k$  is a positive integer; and let  $\mathcal{G} = \{f^{(k)} : f \in \mathcal{F}\}$ . If  $\mathcal{G}$  is normal in  $\Omega$ , then  $\mathcal{F}$  is also normal in  $\Omega$ .*

### 3. Proof of Theorems

**Proof of Theorem 1.4.** Since normality is a local property, it is enough to show that  $\mathcal{F}$  is normal at each  $z_0 \in \Omega$ . We distinguish the following cases.

**Case 1.**  $\psi_1(z_0), \psi_2(z_0), \psi_3(z_0)$  are distinct.

We further consider following subcases.

**Case 1.1.** Suppose that there exists  $f \in \mathcal{F}$  such that  $f(z_0) \neq \psi_i(z_0)$  for  $i = 1, 2, 3$ . Then we can find a small neighborhood  $D_r(z_0)$  in  $\Omega$  such that  $f(z) \neq \psi_i(z)$  for  $i = 1, 2, 3$  in  $D_r(z_0)$ . By the hypothesis we see that for every  $f(z) \in \mathcal{F}$ ,  $f(z) \neq \psi_i(z)$  for  $i = 1, 2, 3$  in  $D_r(z_0)$ . Thus by Theorem 1.2,  $\mathcal{F}$  is normal at  $z_0$ .

**Case 1.2.** Suppose that there exists  $f \in \mathcal{F}$  such that  $f(z_0) = \psi_i(z_0)$  for  $i = 1$  or 2 or 3. Without loss of generality we assume  $f(z_0) = \psi_2(z_0) = 0$  and  $\psi_3(z_0) = \infty$ . Since  $\psi_i(z_0) \neq \psi_j(z_0)$  ( $1 \leq i < j \leq 3$ ), we can find a small neighborhood  $D_r(z_0)$  in  $\Omega$  such that  $f(z) \neq \psi_i(z)$  for  $i = 1, 2, 3$  in  $D'_r(z_0)$  and  $\psi_1(z) \neq 0, \infty$  ( $\psi_1 \not\equiv 0$ ) in  $D_r(z_0)$ . By the hypothesis we see that for every  $f \in \mathcal{F}$ ,  $f(z) \neq \psi_i(z)$  for  $i = 1, 2, 3$  in  $D'_r(z_0)$ . Thus by Theorem 1.2,  $\mathcal{F}$  is normal in  $D'_r(z_0)$ . Now we claim that  $\mathcal{F}$  is normal at  $z_0$ .

We set

$$\mathcal{G} := \{g(z) = f(z) - \psi_1(z) : f \in \mathcal{F}\}.$$

Note that  $\mathcal{F}$  is normal if and only if  $\mathcal{G}$  is normal. Since  $\mathcal{F}$  is normal in  $D'_r(z_0)$ , so  $\mathcal{G}$  is normal in  $D'_r(z_0)$ . Thus, for a sequence  $\{g_n\} \subset \mathcal{G}$ , there exists a subsequence  $\{g_{n_k}\}$  of  $\{g_n\}$  which converges locally uniformly in  $D'_r(z_0)$  to a meromorphic function  $h$ . We consider the following cases.

**Case 1.2.1.** Suppose that  $h \equiv \infty$ . Then

$$\frac{1}{g_{n_k}(z)} \rightarrow 0 \text{ for } z \in \partial D_{\frac{r}{2}}(z_0)$$

and since, for any sequence  $\{f_n\} \subset \mathcal{F}$ ,  $f_n(z)$  omits  $\psi_1(z)$ , we have

$$g_{n_k}(z) \neq 0 \text{ for } z \in D_{\frac{r}{2}}(z_0).$$

Hence there exists  $k_0 > 0$  such that

$$\left| \frac{1}{g_{n_k}(z)} \right| \leq M, \quad \forall k \geq k_0, \quad z \in \partial D_{\frac{r}{2}}(z_0),$$

where  $M > 0$  is a constant. Thus by Maximum modulus principle, we conclude that

$$\left| \frac{1}{g_{n_k}(z)} \right| \leq M, \quad \forall k \geq k_0, \quad \forall z \in D_{\frac{r}{2}}(z_0).$$

It follows that  $\{1/g_{n_k}(z)\}$  converges locally uniformly to 0 in  $D_{r/2}(z_0)$  and hence  $\{g_{n_k}\}$  converges locally uniformly to  $h$  in  $D_{r/2}(z_0)$ . Thus  $\mathcal{G}$  is normal at  $z_0$ .

**Case 1.2.2.** Suppose that  $h \neq \infty$ . Then again there exists an index  $k_0 > 0$  such that

$$|g_{n_k}(z)| \leq M, \quad \forall k \geq k_0, \quad z \in \partial D_{\frac{r}{2}}(z_0)$$

and since, for any sequence  $\{f_n\} \subset \mathcal{F}$ ,  $f_n(z) \neq \infty$  and  $\psi_1(z) \neq \infty$ , we have

$$g_{n_k}(z) \neq \infty \text{ for } z \in D_{\frac{r}{2}}(z_0),$$

where  $M > 0$  is a constant. Thus by Maximum modulus principle, we conclude that

$$|g_{n_k}(z)| \leq M, \quad \forall k \geq k_0, \quad \forall z \in D_{\frac{r}{2}}(z_0).$$

It follows that  $\{g_{n_k}(z)\}$  converges locally uniformly to  $h$  in  $D_{r/2}(z_0)$ . Hence there exists a subsequence of  $\{g_n(z)\}$  which converges locally uniformly to  $h$  in  $D_{r/2}(z_0)$ . Therefore  $\mathcal{G}$  is normal at  $z_0$ .

Thus,  $\mathcal{F}$  is normal at  $z_0$ .

**Case 2.** Exactly two of  $\psi_1(z_0), \psi_2(z_0), \psi_3(z_0)$  are equal.

Without loss generality we assume that  $\psi_1(z_0) = \psi_2(z_0)$  and  $\psi_3(z_0) \neq \psi_1(z_0), \psi_2(z_0)$ . Then, by hypothesis, for every  $f \in \mathcal{F}$ ,  $f(z_0) \neq \psi_i(z_0)$  for  $i = 1, 2$ . We consider the following two subcases.

**Case 2.1.**  $\psi_1(z_0)$  is finite.

Then we can find a small neighborhood  $D_r(z_0)$  in  $\Omega$  such that  $\psi_i(z) \neq \psi_j(z)$  ( $1 \leq i < j \leq 3$ ) in  $D'_r(z_0)$ . Thus by Case 1,  $\mathcal{F}$  is normal in  $D'_r(z_0)$ . Now we turn to show that  $\mathcal{F}$  is also normal at  $z_0$ . Since  $\psi_1(z_0)$  and  $\psi_2(z_0)$  are finite and for every  $f \in \mathcal{F}$ ,  $f(z_0) \neq \psi_i(z_0)$  for  $i = 1, 2$ , we can find that for every  $f \in \mathcal{F}$ ,  $f(z) \neq \psi_i(z)$  for  $i = 1, 2$  and  $\psi_1(z), \psi_2(z)$  are holomorphic in  $D_r(z_0)$ . Then by Lemma 2.2,  $\mathcal{F}$  is normal at  $z_0$ .

**Case 2.2.**  $\psi_1(z_0)$  is infinite.

Then we can find a small neighborhood  $D_r(z_0)$  in  $\Omega$  such that  $\psi_1(z)$  and  $\psi_2(z)$  are holomorphic in  $D'_r(z_0)$  and  $\psi_i(z) \neq 0$  for  $i = 1, 2$  in  $D_r(z_0)$ .

We set

$$\mathcal{G} = \left\{ g = \frac{1}{f} : f \in \mathcal{F} \right\}$$

and

$$\phi_i(z) = \frac{1}{\psi_i(z)} \text{ for } i = 1, 2, 3.$$

Then  $\phi_1(z_0) = \phi_2(z_0) = 0$  and  $\phi_3(z_0) \neq 0$ . Thus, as in Case 2.1, we can prove that  $\mathcal{G}$  is normal at  $z_0$  and hence  $\mathcal{F}$  is normal at  $z_0$ .

**Case 3.**  $\psi_1(z_0) = \psi_2(z_0) = \psi_3(z_0)$ .

By the hypothesis, we have for every  $f \in \mathcal{F}$ ,  $f(z_0) \neq \psi_i(z_0)$  for  $i = 1, 2, 3$ . Then we can find a small neighborhood  $D_r(z_0)$  in  $\Omega$  such that for every  $f \in \mathcal{F}$ ,  $f(z) \neq \psi_i(z)$  for  $i = 1, 2, 3$  in  $D_r(z_0)$ . Hence by Theorem 1.2,  $\mathcal{F}$  is normal at  $z_0$ .

This completes the proof of Theorem 1.4. ■

**Proof of Theorem 1.9.** Since normality is a local property, it is enough to show that  $\mathcal{F}$  is normal at each  $z_0 \in \Omega$ . We distinguish the following cases.

**Case 1.**  $\psi_i(z_0) \neq \psi_j(z_0)$  for  $1 \leq i < j \leq 3$ .

Then we can find a small neighborhood  $D_r(z_0)$  in  $\Omega$  such that  $\psi_i(z) \neq \psi_j(z)$  ( $1 \leq i < j \leq 3$ ) in  $D_r(z_0)$ . Since, for every  $f, g \in \mathcal{F}$ ,  $f^{(k)}$  and  $g^{(k)}$  share the set  $S = \{\psi_1(z), \psi_2(z), \psi_3(z)\}$ . Thus by Theorem 1.4,  $\{f^{(k)} : f \in \mathcal{F}\}$  is normal in  $D_r(z_0)$ , so by Lemma 2.3,  $\mathcal{F}$  is normal in  $D_r(z_0)$ . Thus  $\mathcal{F}$  is normal at  $z_0$ .

**Case 2.**  $\psi_i(z_0) = \psi_j(z_0)$  for  $1 \leq i < j \leq 3$ .

Proceeding in the same way as in Case 2 and Case 3 of Theorem 1.4, we conclude that  $\mathcal{F}$  is normal at  $z_0$ .

This completes the proof of Theorem 1.9. ■

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